

On the Theoretical Values of the Secular Accelerations in the Lunar Theory. By Ernest W. Brown.

(Abstract.)

The accurate determination of certain inequalities in the Moon's motion has made it possible to find the theoretical values of the secular accelerations with a high degree of accuracy. The method of determination consists in using the theorem of Professor Newcomb for finding the secular accelerations in combination with that of the author connecting the mean motions of the perigee and node with the constant part of the lunar parallax. The resulting values, with the greatest possible amounts by which they are uncertain (assuming Newcomb's value for $\delta e'$ to be correct), are :

Secular accelerations per century of the

Mean motion	+ 5''91	± 0''02,
Perigee	- 38 9	± 0'1,
Node	+ 6 56	± 0'02.

1. The theoretical determination of the secular accelerations, formerly one of the most difficult and tedious problems in the lunar theory, has been rendered quite simple by the researches of Professor Newcomb.* Until the publication of these, two years ago, the only investigation with any approach to high accuracy was that of Delaunay, who carried the approximations so far as to include a great number of terms. The enormous labour of finding these, however, was hardly compensated by the result obtained, owing to the uncertainty produced by the slow convergence of the series when arranged according to powers of m . Hansen's results are entirely numerical, and he used those furnished by observation in the construction of his tables. In Newcomb's memoir, just referred to, the values are deduced from Delaunay's algebraical expressions as obtained in the *Théorie de la Lune*, but these again suffer from slow convergence, and errors must occur, first, from incorrect estimations of the remainders, and, secondly, from errors in Delaunay's theory. Owing to the former cause, Professor Newcomb concludes that his value ($5''\cdot60$) for the secular acceleration of the mean motion is uncertain by more than $\frac{1}{20}$ of the whole. He adds (mem. cit. p. 199) : "Although all the observations hitherto made on the Moon would not enable us to detect errors of this magnitude, a more accurate determination would be desirable from a theoretical point of view." It is the

* Papers published for the use of the *Amer. Eph.* vol. v. pt. iii.

object of this paper to show how such a determination may be made, and to find the values correctly to $\frac{1}{300}$ part of the whole in each case.

This has been rendered possible in three ways: First, by Newcomb's theorem, showing how Delaunay's final constants L , G , H may be calculated from the expressions for the rectangular coordinates of the Moon; secondly, by my theorem connecting the constant term in the expression for the parallax with L , G , H and the motions of the perigee and node;* and, thirdly, by the new determinations of these mean motions, the results of which have been given in my paper "On the Mean Motions of the Lunar Perigee and Node."† These theorems have also been employed to test Delaunay's algebraical expressions for the secular accelerations, and the results are given in § 4 below. In order to find the numerical values, such expressions as $\frac{\partial^2 \pi}{\partial n \partial .e'^2}$ have to be calculated; it is shown in § 5 how they may be found with very fair accuracy when the complete numerical value of $\frac{\partial \pi}{\partial e'^2}$ and only a few terms of its slowly convergent algebraical expression are known. Finally, the numerical results are given, with the amounts by which they are uncertain.

General Formulæ for the Secular Accelerations.

2. Professor Newcomb's theorem (mem. cit. p. 191) states that the variations of n (or a), e , γ due to a variation $\delta e'$ of e' may be obtained accurately to the first order, with respect to δ , by solving the equations

$$\delta L = 0, \quad \delta G = 0, \quad \delta H = 0,$$

or

$$\frac{\partial L}{\partial n} \delta n + \frac{\partial L}{\partial .e^2} \delta .e^2 + \frac{\partial L}{\partial .\gamma^2} \delta .\gamma^2 = - \frac{\partial L}{\partial .e'^2} \delta .e'^2, \text{ \&c.};$$

and that the secular accelerations of the mean motion, the perigee and node, are then given by

$$\delta \epsilon = \int \delta n \, dt, \quad \delta \pi = \int \delta \pi_1 \, dt, \quad \delta \theta = \int \delta \theta_1 \, dt,$$

where π_1 , θ_1 are the mean motions of the perigee and node. Putting $L = c_1$, $G - L = c_2$, $H - G = c_3$, the first three equations become

$$\delta c_1 = 0, \quad \delta c_2 = 0, \quad \delta c_3 = 0.$$

* *Proc. London Math. Soc.* 1897.

† *Monthly Notices R.A.S.* 1897.

I have shown in the latter of the two papers just referred to that the variations of n , e , γ , e' also satisfy the equations

$$\begin{aligned} c_2 \delta \cdot \frac{\partial \pi_1}{\partial n} + c_3 \delta \cdot \frac{\partial \theta_1}{\partial n} &= \frac{3}{2} (E + M) \delta \cdot \frac{\partial}{\partial n} \left(\frac{1}{r} \right)_0, \\ c_2 \delta \cdot \frac{\partial \pi_1}{\partial \cdot e^2} + c_3 \delta \cdot \frac{\partial \theta_1}{\partial \cdot e^2} &= \frac{3}{2} (E + M) \delta \cdot \frac{\partial}{\partial \cdot e^2} \left(\frac{1}{r} \right)_0, \\ c_2 \delta \cdot \frac{\partial \pi_1}{\partial \cdot \gamma^2} + c_3 \delta \cdot \frac{\partial \theta_1}{\partial \cdot \gamma^2} &= \frac{3}{2} (E + M) \delta \cdot \frac{\partial}{\partial \cdot \gamma^2} \left(\frac{1}{r} \right)_0. \end{aligned}$$

These results immediately follow from those given in my former paper in the *Monthly Notices* by submitting them to a variation δ and putting δc_1 , δc_2 , δc_3 zero. As before, $E + M = n^2 a^3$, and $\left(\frac{1}{r} \right)_0$ denotes the constant term in the expression of $\frac{1}{r}$; the constants are defined as in Delaunay's final expressions for the coordinates. The square of the ratio of the solar to the lunar parallax is neglected, but its effect is quite insensible.

The first of these three equations may be used effectively to find the secular accelerations. We have

$$\delta \cdot \frac{\partial \pi_1}{\partial n} = \frac{\partial^2 \pi_1}{\partial n^2} \delta n + \frac{\partial^2 \pi_1}{\partial n \partial \cdot e'^2} \delta \cdot e'^2 + \frac{\partial^2 \pi_1}{\partial n \partial \cdot e^2} \delta \cdot e^2 + \frac{\partial^2 \pi_1}{\partial n \partial \cdot \gamma^2} \delta \cdot \gamma^2,$$

with similar results for the other expressions. It can be easily shown that c_2 , $\delta \cdot e^2$ contain the factor e^2 in all their terms, and c_3 , $\delta \cdot \gamma^2$ the factor γ^2 . Also it is known by a theorem of Adams * that there are no terms in $\left(\frac{1}{r} \right)_0$ of the form $\frac{1}{a} e^2 f(m, e'^2)$,

$\frac{1}{a} \gamma^2 f(m, e'^2)$. Hence, in order to find δn to the orders e^2 , γ^2 , and all powers of e'^2 , we can neglect $\delta \cdot e^2$, $\delta \cdot \gamma^2$ in the first equation, so that it becomes

$$\begin{aligned} c_2 \left(\frac{\partial^2 \pi_1}{\partial n^2} \delta n + \frac{\partial^2 \pi_1}{\partial n \partial \cdot e'^2} \delta \cdot e'^2 \right) + c_3 \left(\frac{\partial^2 \theta_1}{\partial n^2} \delta n + \frac{\partial^2 \theta_1}{\partial n \partial \cdot e'^2} \delta \cdot e'^2 \right) \\ = \frac{3}{2} (E + M) \left[\frac{\partial^2 P_0}{\partial n^2} \delta n + \frac{\partial^2 P_0}{\partial n \partial \cdot e'^2} \delta e'^2 \right], \end{aligned}$$

where P_0 has been put for $\left(\frac{1}{r} \right)_0$.

If we neglect e^2 , γ^2 , it is seen that the principal part of δn is given by

$$\frac{\partial^2 P_0}{\partial n^2} \delta n = - \frac{\partial^2 P_0}{\partial n \partial \cdot e'^2} \delta e'^2 \dots \dots \dots (I)$$

* *Monthly Notices*, vol. xxxviii. pp. 460-472. *Coll. Works*, vol. i. pp. 189-204.

When the principal part of δn has been found from this equation, its value may be substituted in the left-hand member of the previous equation, and the parts depending on e^2 , γ^2 can then be evaluated.

For the motion of the perigee we have

$$\delta\pi_1 = \frac{\partial\pi_1}{\partial e'^2} \delta \cdot e'^2 + \frac{\partial\pi_1}{\partial n} \delta n + \frac{\partial\pi_1}{\partial e^2} \delta \cdot e^2 + \frac{\partial\pi_1}{\partial \gamma^2} \delta \cdot \gamma^2.$$

If we neglect e^2 , γ^2 , it will be sufficient to use the first two terms only. The parts of chief importance in all cases arise from $\partial\pi_1/\partial e'^2$, and their accuracy depends mainly on the accuracy with which the terms having the factor e'^2 in π_1 are known. If, however, the parts in $\delta\pi_1$, depending on e^2 , γ^2 , be required, it will be simplest to find $\delta \cdot e^2$, $\delta \cdot \gamma^2$ from the equations $\delta c_2 = 0$, $\delta c_3 = 0$, respectively. The principal parts of $\delta \cdot e^2$, $\delta \cdot \gamma^2$ are by this method found in the forms

$$n \delta \cdot e^2 = e^2 f(m) \delta n, \quad n \delta \cdot \gamma^2 = \gamma^2 f(m) \delta n,$$

and can therefore be found, since the value of δn is already known.

In the present state of the lunar theory nothing is to be gained by calculating these higher portions in the case of the perigee, for the complete numerical values of the terms in π_1 , which have the factors $e^2 e'^2$, $\gamma^2 e'^2$ have not yet been obtained, and Adams' value of P_0 furnishes only the terms up to the orders $e^2 e'^2 m^3$, $\gamma^2 e'^2 m^3$. Hence we must verify Delaunay's values to this order and estimate for the remainders. Fortunately, owing to the smallness of the quantities involved, the possible errors of estimation are very small and can be easily seen.

Exactly similar results hold for $\delta\theta_1$.

3. The equation (1) explains partly the slowness of the convergence of the expression which represents the principal part of δn when expanded in powers of m . All inequalities in the lunar theory which contain the factor e' appear to be more or less slowly convergent. When such expressions as $e'^2 f(m)$ are differentiated with respect to n , the earlier terms will converge more slowly still, but the convergence of the later terms will not be much affected. The calculations further show that slow convergence in the higher terms will arise partly from the expansion of the inverse of $\frac{\partial^2 P_0}{\partial n^2}$ in powers of m ; a remedy for this is evident. The convergence of all the results is, as usual, improved by using $m_1 = n'/(n - n')$ instead of $m = n'/n$ as a parameter.

4. *A Test of Delaunay's Expressions for the Secular Accelerations.*—These are given in *C.R.* vol. xlv. p. 209, in terms of Plana's constants, and, with an increased number of terms, in *C.R.* vol. lxxiv. p. 174, in terms of his own constants. The constant part

of $1/r$ has been obtained by Adams (*Coll. Works*, vol. i. p. 203), and his value was used in the formulæ of § 2. The values of c_2, c_3 are given by Newcomb (mem. cit. p. 184), and the values of π_1, θ_1 are those of Delaunay (*C.R.* vol. lxxiv. p. 19) with the corrections noted in my paper in the *Monthly Notices* already referred to (§ 1). The last powers of m tested in each case are as follows :—

The value of $\delta\epsilon$ was tested as far as $m^7, \gamma^2 m^5, e^2 m^5, e'^2 m^5$, and the values of $\delta\pi, \delta\theta$ as far as $m^5, \gamma^2 m^3, e^2 m^3, e'^2 m^2$. The only differences found were the coefficients of the terms involving $\gamma^2 m^3$ in $\delta\pi$ and $e^2 m^3$ in $\delta\theta$, for which $-\frac{694}{4}, -\frac{694}{16}$ were found instead

of $-\frac{693}{4}, -\frac{693}{16}$. The terms involving a^2/a'^2 and higher powers and products of e^2, γ^2, e'^2 do not produce additions so great as $0''.01$, and they will therefore not be considered.

5. *Method of combining algebraical and numerical results to obtain the derivatives with regard to n.*—The problem may be stated as follows : Given the first i terms and also the complete numerical value of a series whose terms are diminishing in absolute value, to find a close approximation to the value of its derivative. The following is a practical method of solution :—

Let

$$f(m) = a_0 + a_1 m + a_2 m^2 + \dots,$$

where a_0, a_1, \dots are numerical coefficients, and put

$$f_i(m) = a_0 + a_1 m + a_2 m^2 + \dots + a_{i-1} m^{i-1}.$$

Taking the derivatives with regard to m , we can write the result

$$mf'(m) = mf'_i(m) + i \left[a_i m^i + \frac{i+1}{i} a_{i+1} m^{i+1} + \dots \right].$$

If i be not too small (*e.g.* not less than 5), and the terms of the series are not diminishing too slowly (*e.g.* with a ratio not much greater than $\frac{1}{2}$), the error committed by putting

$$mf'(m) = mf'_i(m) + i[mf(m) - mf_i(m)]$$

will be small compared with the true value of $mf'(m)$. We shall generally obtain greater accuracy by using $i +$ a proper fraction instead of i as the factor, when the series appears to be diminishing with fair regularity.

For example, the part of π_1 which depends on e'^2 when the parameter m_1 is used, is

$$ne'^2 \left(\frac{9}{8} m_1^2 + \frac{753}{32} m_1^3 + \frac{42243}{256} m_1^4 + \frac{942777}{1024} m_1^5 \right).$$

Here $i=6$, and reducing the portion in brackets to numbers, we obtain

$$f_6(m_1) = +.00735 + .01244 + .00705 + .00318 = +.03002.$$

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I find

$$f(m_1) = +0.32063.$$

Hence

$$f'(m_1) - f_6(m_1) = +0.00204.$$

Applying the formula, and using $6\frac{1}{2}$ as the multiplying factor of this difference, we find

$$f'(m_1) = +0.147 + 0.373 + 0.282 + 0.159 + 0.133 = +1.094.$$

The result might possibly be as much as four or five units wrong in the last place of decimals, but in all the cases treated below, the possible error does not affect the results to the degree of accuracy required. To complete the determination, we have, for this portion of π_1 ,

$$\begin{aligned} \frac{\partial \pi_1}{\partial n} &= f(m_1) - (1 + n_1)m_1 f'(m_1) \\ &= +0.321 - 1.182 = -0.86, \end{aligned}$$

which is not more than one unit wrong in the third place of decimals.

The Numerical Values.

6. The following are the numerical data required :—

In the values of c_2, c_3 , we only require the terms which are of the forms $na^2e^2f(m)$, $na^2\gamma^2f(m)$, respectively. These may be accurately obtained by Newcomb's method from the expressions in terms of rectangular coordinates, c_3 from the results in a paper by Mr. P. H. Cowell,* and c_2 from those in a paper by the writer.† We find

$$c_2 = -0.474035na^2e^2, \quad c_3 = -2.00257na^2\gamma^2.$$

The values of π_1, θ_1 are as follows :‡

$$\begin{aligned} \pi_1 &= n[+0.0085726 + 0.32063e'^2], \\ \theta_1 &= n[-0.039992 - 0.005224e'^2]. \end{aligned}$$

Proceeding as in § 5, we then find

$$\begin{aligned} \frac{\partial \pi_1}{\partial n} &= -0.1502 - 0.86e'^2, \\ \frac{\partial \theta_1}{\partial n} &= +0.03708 + 0.00319e'^2, \\ n \frac{\partial^2 \pi_1}{\partial n^2} &= -0.0489, & n \frac{\partial^2 \theta_1}{\partial n^2} &= -0.00661. \end{aligned}$$

* *Amer. Jour. Math.* vol. xviii. p. 113.

† *Ibid.* vol. xv. p. 261.

‡ See the paper "On the Mean Motions of the Lunar Perigee and Node."

The numbers contain powers of m only ; the omitted portions, depending on e^2 , etc., are not required.

Further, from Adams' algebraical results and my numerical calculations :

$$\begin{aligned} aP_0 &= 1.00091 + .0011750e'^2 + .00064e'^4, \\ an \frac{\partial P_0}{\partial n} &= +.66551 - .0009865e'^2 + .0026e'^4, \\ an^2 \frac{\partial P_0}{\partial n^2} &= -.21975 - .0001315e'^2. \end{aligned}$$

7. Pursuing the operations of § 2 above, we obtain

$$\begin{aligned} \delta n &= [-.004489 + .024e'^2 + .1235e^2 - .0196\gamma^2]n\delta \cdot e'^2, \\ \delta\pi_1 &= +.03213n\delta \cdot e'^2, \\ \delta\theta_1 &= -.005241n\delta \cdot e'^2. \end{aligned}$$

Put now, with Delaunay,

$$\int n\delta e'^2 dt = -1270''T^2,$$

where T is the time reckoned in centuries of Julian years, and include the parts of $\delta\pi_1$, $\delta\theta_1$ which depend on e^2 , γ^2 , e'^2 from his expressions (see § 4 above), estimating for the remainders in each case. The maximum possible error due to the latter is appended, with the resulting values in the table :

Secular Accelerations.			
Parts of form.	In δn .	In $\delta\pi_1$.	In $\delta\theta_1$.
$f(m)$	+ 5.70	- 40.8	+ 6.66
$e'^2 f(m)$	- 0.01	+ 0.0	- 0.00
$e^2 f(m)$	+ 0.47	+ 0.2	+ 0.14
$\gamma^2 f(m)$	- 0.05	+ 0.4	- 0.02
Complete values	+ 6.11 \pm 0.02	- 40.2 \pm 0.1	+ 6.78 \pm 0.02

These results agree almost exactly with those of Delaunay. With the latest values for the masses of the planets, Professor Newcomb finds (mem. cit. p. 199),

$$\int n\delta e'^2 = -1229''T^2.$$

Hence the definitive theoretical values of the secular accelerations per century are, for

The Mean Motion	+ 5.91 \pm 0.02;
The Perigee	- 38.9 \pm 0.1;
The Node	+ 6.56 \pm 0.02.

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Laying aside the question of the accuracy of the value of $\int n\delta e'^2$, the above theoretical values for the secular acceleration are correct to $\frac{1}{300}$ part of the whole in each case.

Haverford College, Pa.:
1897 January 11.

On a Photographic Transit Circle. By H. H. Turner, M.A.,
B.Sc., Savilian Professor.

1. The desirability of introducing photography into meridian observations, as a method of eliminating personal and other errors, has been recognised by many astronomers, and will probably be admitted by all, if this introduction can be accomplished without endangering the simplicity and stability of the present form of instrument. The following solution of this problem, which presents some advantages not realised in other forms, is offered for consideration.

2. It may be well in the first instance to enumerate the chief features of the present instrument—its advantages and disadvantages.

Features:

- (a) Movable visual telescope with two micrometers at right angles.
- (b) Two carefully made pivots, allowing one degree of freedom only.
- (c) Graduated circle and microscopes.
- (d) Chronograph.

All these, or their equivalents, may thus be allowed in any new arrangement.

Advantages:

- (e) Stability, owing to there being only one degree of freedom, excepting in so far as the micrometers are used.
- (f) Simplicity.

The advantages may generally be brought under one or other of these heads. It is unnecessary to specify them in detail unless we are going to propose any rearrangement of the instrument which will sacrifice any of them, and that is not my present purpose. It is more important to consider carefully how we may remove the

Disadvantages of the Present Instrument, viz.:

- (g) Personal equation of all kinds.
- (h) Flexure and irregular refraction.
- (i) Limitation to comparatively bright stars.

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